

Weak lensing and the Dyer–Roeder approximation

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ABSTRACT

The distance-redshift relation plays an important role in cosmology. In the standard approach to cosmology it is assumed that this relation is the same as in the homogeneous universe. As the real universe is not homogeneous there are several methods to calculate the correction. The weak lensing approximation and the Dyer–Roeder relation are one of them. This paper establishes a link between these two approximations. It is shown that if the universe is homogeneous with only small, vanishing after averaging, density fluctuations along the line of sight, then the distance correction is negligible. It is also shown that a vanishing 3D average of density fluctuations does not imply that the mean of density fluctuations along the line of sight is zero. In this case, even within the linear approximation, the distance correction is not negligible. The modified version of the Dyer–Roeder relation is presented and it is shown that this modified relation is consistent with the correction obtained within the weak lensing approximation. The correction to the distance for a source at $z \sim 2$ is of order of a few percent. Thus, with an increasing precision of cosmological observations an accurate estimation of the distance is essential. Otherwise errors due to miscalculation the distance can become a major source of systematics.

Key words: gravitational lensing; weak – cosmology: theory – large-scale structure of Universe

1 INTRODUCTION

The distance-redshift relation plays an important role in cosmology. In fact almost all cosmological observations depend, either explicitly or implicitly, on this relations. However, in general case, to calculate the distance, knowledge of matter distribution (as well as its evolution) between the observer and the source is needed. Therefore, an approximate relations are of great use to astronomers. The simplest approximation assumes homogeneity of the universe and is based on the Friedmann model. However, as the real universe is not homogeneous a more elaborate relations are needed. Zel’dovich (1964) proposed an approximation which takes into account that light propagates through emptier rather than denser regions of the universe. This approximation is now known as the Dyer–Roeder approximation (Dyer & Roeder 1972, 1973). It also assumes homogeneity but allows for a different density than in the background model. The density difference is modelled by a constant, the so-called smoothness parameter α . As the Dyer–Roeder equation is a differential equation an approximate analytic solution was presented and discussed by Demiański et al. (2003). Generalisation to $\alpha(z)$ was first suggested by Linder (1988) and the effect of the change of the expansion rate was discussed by Mattsson (2010). The Dyer–Roeder relation was tested

against cosmological observations by Santos & Lima (2006), Santos, Cunha & Lima (2008), and Yu et al. (2010).

However, it has been argued that the Dyer–Roeder relation may not properly describe the effect of matter clustering (Räsänen 2009; Ellis 2009) and therefore may not be an appropriate approximation for the distance-redshift relation in the real universe. Also, if the smoothness parameter changes with redshift the distance formula depends on several free-parameters whose interpretation can be ambiguous. Therefore, although cosmologists are aware that not taking into account inhomogeneities introduce additional systematics, the Dyer–Roeder relation is not widely used. Instead, several alternatives based either on the linear perturbative scheme (see Bonvin, Durrer & Gasparini 2006 and references therein) or non-linear models (see Bolejko et al. 2009 and references therein) have been developed.

This paper explores the connection between the Dyer–Roeder approximation and the weak lensing approximation. It shows that the Dyer–Roeder relation can be modified so that it is consistent with the lensing approximation. The modified version contains only 1 free parameter of clear interpretation, i.e. the mean of density fluctuations along the line of sight $\langle \delta \rangle_{1D}$.

The structure of this paper is as follows: Sections 2 and 3 present the Dyer–Roeder and weak lensing approximations

respectively, Sec. 4 presents the comparison of these two methods, and Sec. 5 discusses the results.

2 THE DYER–ROEDER APPROXIMATION

The angular diameter distance D_A is given by the following relation (Sachs 1961)

$$\frac{d^2 D_A}{ds^2} = -(|\sigma|^2 + \frac{1}{2} R_{\alpha\beta} k^\alpha k^\beta) D_A, \quad (1)$$

where σ is the shear of the light bundle, k^α is a vector tangent to the light ray, $R_{\alpha\beta}$ is the Ricci tensor, and $R_{\alpha\beta} k^\alpha k^\beta = \kappa T_{\alpha\beta} k^\alpha k^\beta$ (where $T_{\alpha\beta}$ is the energy-momentum tensor). In the comoving and synchronous coordinates, for pressure-less matter, $T_{\alpha\beta} k^\alpha k^\beta = \kappa \rho k^0 k^0$. The Dyer–Roeder approach assumes homogeneity ($\sigma = 0 = \delta\rho$) but takes into account that light propagates through vacuum. Therefore, Ω_m which photons ‘feel’ is different than true Ω_m . This is modelled by a constant parameter α (of value between 0 and 1) the multiplies Ω_m . In this case (1) reduces to

$$\frac{d^2 D_A}{dz^2} + \left(\frac{1}{H} \frac{dH}{dz} + \frac{2}{1+z} \right) \frac{dD_A}{dz} + \frac{3\alpha\Omega_m H_0^2}{2H^2} (1+z) D_A = 0, \quad (2)$$

where $H(z) = H_0 \sqrt{\Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda}$. The initial conditions needed to solve (2) are: $D_A = 0$ and $dD_A/dz = 1/H_0$.

Generalization to $\alpha(z)$ was explored by Linder (1988) who suggested several algebraic forms like $\alpha(z) = \alpha_0 + \alpha_1 z$ or $\alpha(z) = \alpha_* + \alpha_2(1+z)^\gamma$. Santos & Lima (2006) proposed $\alpha(z) = \beta_0(1+z)^{3\gamma}/[1 + \beta_0(1+z)^{3\gamma}]$. This paper studies the following form of $\alpha(z)$

$$\alpha(z) = 1 + \mathcal{D}(z)\langle\delta\rangle_{1D} \quad (3)$$

where $\langle\delta\rangle_{1D}$ is the mean of present-day density fluctuations along the line of sight, and $\mathcal{D}(z)$ describes its evolution. Below it is shown that if $\mathcal{D} = (1+z)^{-5/4}$, then the Dyer–Roeder equation gives results consistent with the results obtain under the assumption of the lensing approximation.

3 LINEAR PERTURBATIONS AND THE LENSING APPROXIMATION

Writing the distance as

$$D_A(z) = \bar{D}_A(1 + \delta_D), \quad (4)$$

where \bar{D}_A the distance in the homogeneous universe, one can derive formula for δ_D using the linear perturbative scheme. The most general form was presented and discussed by Pyne & Birkinshaw (2004), Bonvin et al. (2006), Hui & Greene (2006), and Enqvist, Mattsson & Rigopoulos (2009). Excluding the contribution from the motion of the observer and source, and taking the leading term, δ_D reduces to

$$\delta_D = - \int_0^{\chi_e} d\chi \frac{\chi_e - \chi}{\chi_e} \chi \nabla^2 \phi(\chi), \quad (5)$$

where χ is the comoving coordinate $d\chi = dz/H(z)$, ϕ is the gravitational potential which can be related to density perturbations $\rho\delta$ via the Poisson equation $\nabla^2 \phi = \frac{4\pi G}{c^2} a^2 \rho\delta$. Equation (5) is equivalent to the convergence in the lensing

approximation and is known as the Born approximation. As seen voids ($\delta < 0$) increase the distance while regions of $\delta > 0$ decrease it.

In order to solve (5) one needs to know $\nabla^2 \phi$ along the line of sight. Using the Poisson equation, the gravitational potential is related to the density fluctuations. Thus when calculating the variance, one can Fourier transform δ and use the matter power spectrum instead (Munshi & Jain 2000, 2001). However, in this paper we are not interested in the variance or higher order moments, just in the distance itself. Therefore, to solve (5) actual density fluctuations along the line of sight are needed.

The preset-day fluctuations are non-linear and therefore no longer Gaussian. Only when density fluctuations are in the linear regime, their probability distribution function (PDF) can be approximated by the Gaussian PDF. Once density fluctuations are in the non-linear regime they are no longer Gaussian (there is no symmetry as $-1 \leq \delta < \infty$). However, it has been shown that in the non-linear regime density fluctuations can be approximated by the one-point log-normal PDF (Kayo, Taruya & Suto 2001; Lahav & Suto 2004)

$$P(\delta) = \frac{1}{\sqrt{2\pi\sigma_{nl}^2}} \exp \left[-\frac{(\ln(1+\delta) + \sigma_{nl}^2/2)^2}{2\sigma_{nl}^2} \right] \frac{1}{1+\delta}, \quad (6)$$

where

$$\sigma_{nl}^2 = \ln[1 + \sigma_R^2] \quad \text{and} \quad \sigma_R^2 = \frac{1}{2\pi^2} \int_0^\infty dk \mathcal{P}(k) W^2(kR) k^2, \quad (7)$$

where $\mathcal{P}(k)$ is the matter power spectrum.

Two methods for generating density fluctuations along the line of sight are considered. Both have a log-normal PDF of δ (3D). The first method ensures that $\langle\delta\rangle_{1D} \approx 0$ (the mean of the present-day fluctuations along the line of sight). The second method, on the other hand, allows for $\langle\delta\rangle_{1D} \neq 0$.

4 RESULTS

4.1 Vanishing mean of the density fluctuations along the line of sight

First let us focus on generating the density fluctuations using directly the log-normal PDF. Using (6) δ smoothed in a sphere of radius R can be generated. Density fluctuations in this model are schematically presented in Fig. 1a – when the light ray exits one sphere it enters another one of different R and δ (for details see Appendix A).

The distance correction δ_D calculated from (5) is presented in Fig. 2. One point about Fig. 2. needs to be emphasized. As follows from (5) the distance correction depends on the position of the source, for example $\delta_D(z = 0.5)$ for a source at $z_* = 2$ is different than $\delta_D(z = 0.5)$ for a source at $z_* = 0.5$ even if density fluctuations along the line of sight are the same. Thus, the distance correction presented in Fig. 2 is the distance correction for the source at given z , i.e. $\delta_D(z = z_*)$. Fig. 2 presents the mean and variance: at each z (a discrete number of z with an interval $\Delta z = 0.01$ was considered) based on 100,000 runs (each with a different distribution of density fluctuations along the line of sight) the mean and variance were calculated. The mean and variance

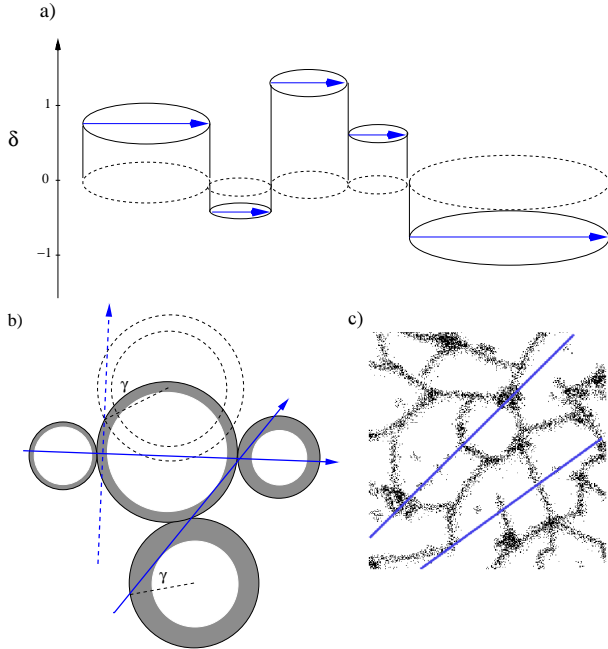


Figure 1. Density inhomogeneities along the line of sight. Panel a): density fluctuations generated from the log-normal PDF (6) – when the light ray exits one structure, the next one is generated. Panel b): compensated density fluctuations – when the light ray exits a structure, another one is generated together with an initial angle at which the ray enters the structure. When γ is large then the light enters and exits the structure at large angles. In this case structures can overlap. Panel c): light propagation in the real universe.

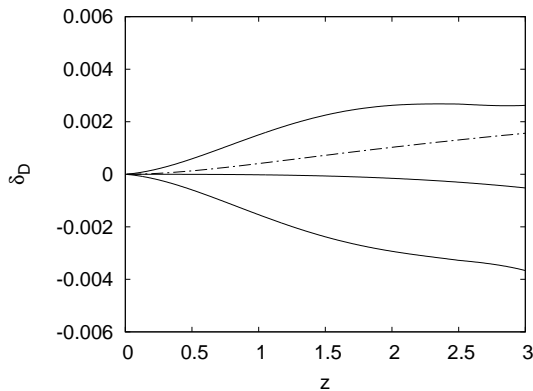


Figure 2. Distance correction δ_D within the weak lensing approximation for a model with a negligible mean of density fluctuations along the line of sight (Sec. 4.1). Dash-dotted line presents δ_D obtained using the Dyer–Roeder relation with the smoothness parameter $\alpha = 0.99$ ($\alpha = 1$ is equivalent to the standard Friedmann relation).

are represented by solid lines. For comparison, the Dyer–Roeder approximation with $\alpha = 0.99$ is also presented (when $\alpha = 1$ the Dyer–Roeder approximation reduces to the standard Friedmann relation for the distance). As seen δ_D is of negligible amplitude $\sim 10^{-3}$. Thus, for this configuration, and under considered assumptions, there is no need to take into account inhomogeneities. Also, another important fact is that δ_D in the weak lensing approximation is negative,

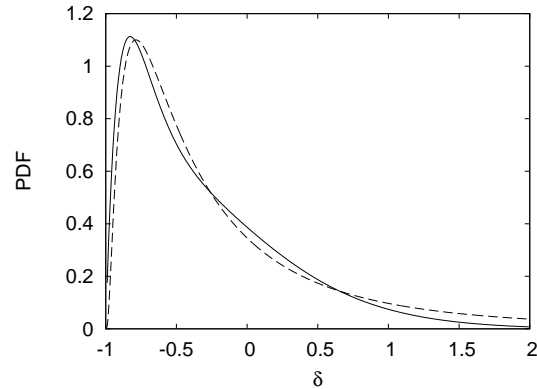


Figure 3. PDF of δ smoothed in a sphere of radius 4 Mpc for a model discussed in Sec. 4.2 (solid line) and the log-normal PDF (6) of δ smoothed also in a sphere of radius 4 Mpc (dashed line).

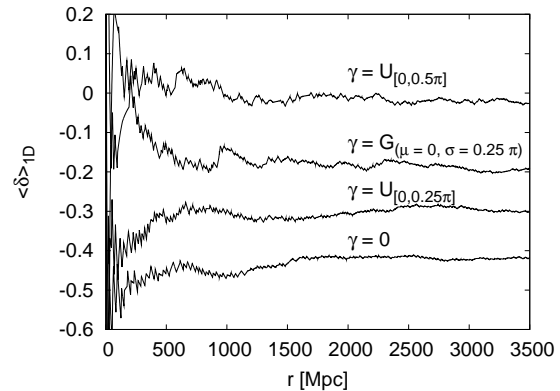


Figure 4. Mean density fluctuations along the line of sight (8) for a model discussed in Sec. 4.2. The upper most curve presents a model with γ generated from a uniform distribution between 0 and 0.5π . The second curve presents a model with γ generated from the Gaussian distribution with the mean equal to zero and the standard deviation equal to 0.25π . In the third model γ is generated from a uniform distribution between 0 and 0.25π . The bottom curve presents a model with $\gamma = 0$.

while the Dyer–Roeder approximation implies positive δ_D . This is a consequence of positive density fluctuations along the line of sight. For example, for a source at $z = 3$ (based on 100,000 runs) the mean of the present-day density fluctuations along the line of sight defined as

$$\langle \delta \rangle_{1D}(r) = \frac{1}{r} \int_0^r d\tilde{r} \delta(t_0, \tilde{r}), \quad (8)$$

is $\langle \delta \rangle_{1D} = 6 \times 10^{-3}$ and the standard deviation 2×10^{-2} , which implies $\delta_D = -5 \times 10^{-4} \pm 3 \times 10^{-3}$. Thus, as pointed out by Räsänen (2009) in order to properly describe the effect of clustering, the smoothness parameter should be larger than 1, not less than 1 as in the Dyer–Roeder approximation. However, in this case the difference as well as the total correction are negligible. However, even if the mean is negligible, the variance can provide additional information – see Linder (2008) for a discussion.

4.2 Non-zero mean of the density fluctuations along the line of sight

In the previous section, random matter fluctuations along the line of sight were considered. In such a case $\langle \delta \rangle_{3D} = 0$ implies $\langle \delta \rangle_{1D} = 0$, and the distance correction is negligible. However, present-day matter fluctuations in the Universe are not purely random but are organized – matter in the Universe forms the cosmic web. In this case $\langle \delta \rangle_{3D} = 0$ does not necessarily imply $\langle \delta \rangle_{1D} = 0$. As seen in Fig 1c the cosmic web contains large voids with fairly compact filaments. Therefore, photons spend more time in voids than in overdense regions. In order to model this phenomenon let us consider a universe that consists of voids surrounded by filaments, each structure compensated. Thus, by the construction a 3D average of density fluctuations is zero (if averaged over sufficiently large scale). Moreover, if the parameters of the system are adjusted (see Appendix B for details) then the PDF of density fluctuations can be almost log-normal. This feature is presented in Fig. 3, which shows the PDF of density fluctuations smoothed within a sphere of radius 4 Mpc. As seen the PDF is similar to log-normal PDF, apart from high δ where PDF is of lower amplitude – but see Jain, Seljak & White (2000) where the PDF of density fluctuations smoothed on scales of $3 h^{-1}$ Mpc has also a lower amplitude for high δ than the log-normal distribution.

When the light ray exits one compensated structure before it enters another one, not only parameters of the next structure are generated, but also an angle γ , at which the light ray enters another structure. Therefore, 4 different methods are going to be considered

- (i) $\gamma = 0$, the light ray always enters another structure at $\gamma = 0$, thus it passes through the centre of the structure.
- (ii) γ is randomly generated from a uniform distribution between 0 and 0.5π . However, as shown in Fig. 1b in this case structures can overlap.
- (iii) to reduce overlapping, γ is generated from the Gaussian distribution with the mean 0 and $\sigma = 0.25\pi$ (if $|\gamma| > 0.5\pi$ then $|\gamma| \rightarrow \pi - |\gamma|$).
- (iv) γ is randomly generated from a uniform distribution between 0 and 0.25π .

The distance correction δ_D (the mean and variance) is presented in Fig. 5. As seen if γ is randomly generated from a uniform distribution between 0 and 0.5π then the mean of density fluctuations along the line of sight is small and δ_D is almost negligible (cf. Brouzakis, Tetrakis & Tzavara 2008; Vanderveld, Flanagan & Wasserman 2008).

However, as pointed out above, if γ changes between 0 and 0.5π then structures can overlap (see Fig 1b). To reduce the overlapping, γ needs to be chosen from a smaller range of angles. In this case δ_D increases. This is because the light rays propagate more likely through voids than filaments.

In Fig. 5 the Dyer–Roeder relation is also presented. The dash-dotted line presents δ_D obtained from the original Dyer–Roeder formula, the dashed line and dotted line present the evolving smoothness parameter (3). Intuitively, if $\langle \delta \rangle_{1D}$ were just an ordinary density perturbation, in the linear regime, it should evolve according to (A1). Thus, the dotted line presents a model with $\alpha(z) = 1 + \delta(z)$, where $\delta(z)$ is given by (A1) (with an initial condition $\delta(z=0) = \langle \delta \rangle_{1D}$).

Following Mattsson (2010) let us consider the Dyer–

Roeder relation with a perturbed expansion rate. From the continuity equation $\dot{\rho} + 3H\rho = 0$, the perturbation in the expansion rate are $\Delta H = -\dot{\delta}/3$ (where δ follows from (A1) with an initial condition $\delta(z=0) = \langle \delta \rangle_{1D}$). We use this relation to calculate the perturbed $H(z)$ and dH/dz , and we then insert it to (2), and using $\alpha(z) = 1 + \delta(z)$, where $\delta(z)$ is given by (A1), we get the distance correction that is presented using the dashed lines.

As seen, neither a constant α , nor the evolving one (with $\delta(z)$ given by (A1)), or with a perturbed expansion rate, are consistent with the weak lensing approximation (except for the case where δ_D is almost negligible). However, if $\alpha(z) = 1 + \langle \delta \rangle_{1D}/(1+z)^\gamma$, where $\gamma = 5/4$ then the Dyer–Roeder approximation produces results comparable with the weak lensing approximation. As seen, the evolution of α is not as intuitively expected, i.e. it does not directly follow from (A1). Here an empirical approach was employed, and it was found that if $\gamma \approx 5/4$ then the modified, in this way, Dyer–Roeder relation leads to the agreement with the lensing approximation.

An important result of the above analysis is that a non-zero mean of density fluctuations along the line of sight can modify the distance by a few percent. As cosmological observations are now reaching the precision of a few percent, a proper handling of the distance is essential. Otherwise the errors in the estimation of the distance can become a major source of systematics. For example a proper handling of distance will be of great importance in the future analysis of BAO experiments (Bolejko 2010).

5 CONCLUSIONS

The distance-redshift relation plays a central role in cosmology. If the homogeneous Friedmann model correctly describes the evolution of the universe on large scales and, in addition, if density fluctuations along the line of sight vanish after averaging, then within the linear approximation, the distance correction is negligible – it is sufficient to apply the Friedmann relation. However, if only the mean of density fluctuations along the line of sight is not zero then, even within the linear approximation, the distance can change by several percent.

In Sec. 4.2 it was shown that a vanishing 3D average of density fluctuations does not imply that the mean of density fluctuations along the line of sight is zero. It is argued that in the real universe this may be the case. In the real universe voids occupy large regions while overdensities are more compact. Moreover, if light propagates for a long time through filaments then it is more likely to be absorbed or scattered. Thus, if a remote galaxy is observed then most likely its photons propagated through emptier rather than denser regions. In this case the weak lensing approximation produces results that are similar to results obtained from the modified version of the Dyer–Roeder equation. The modified relation is

$$\frac{d^2 D_A}{dz^2} + \left(\frac{1}{H} \frac{dH}{dz} + \frac{2}{1+z} \right) \frac{dD_A}{dz} + \frac{3H_0^2}{2H^2} \Omega_m (1+z) \left(1 + \frac{\langle \delta \rangle_{1D}}{(1+z)^{5/4}} \right) D_A = 0, \quad (9)$$

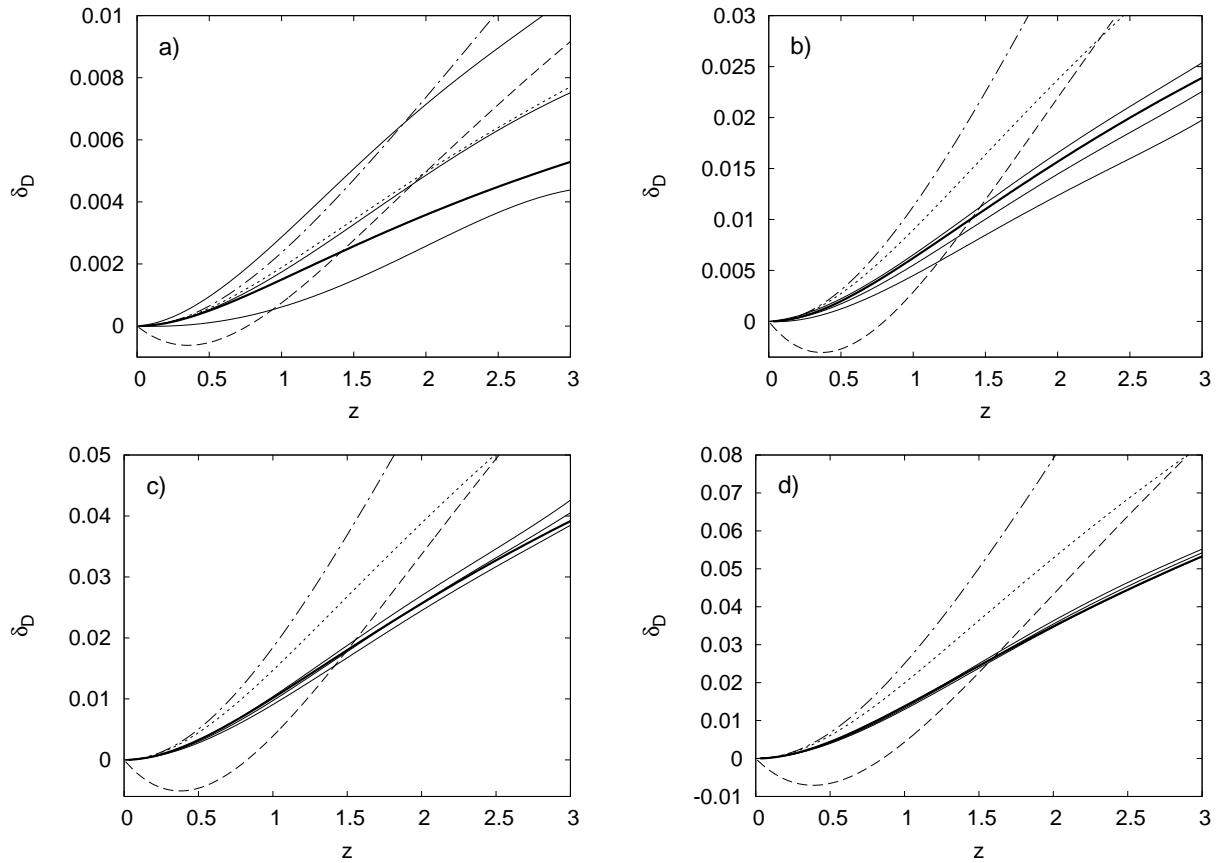


Figure 5. Distance correction δ_D for a model discussed in Sec. 4.2. The weak lensing approximation (5) is presented by thin solid curves (mean + variance), the dash-dotted lines present the Dyer–Roeder approximation with a constant smoothness parameter $\alpha = 1 + \langle\delta\rangle_{1D}$, the dotted lines represent models with $\alpha(z) = 1 + \delta(z)$, where $\delta(z)$ is given by (A1), the dashed lines present the distance correction for models with a perturbed expansion rate (see text for details), and the thick solid lines represent the modified Dyer–Roeder relations with $\alpha = 1 + \langle\delta\rangle_{1D}/(1+z)^{5/4}$. Panel a): γ generated from a uniform distribution between 0 and 0.5π , $\langle\delta\rangle_{1D} \approx -0.04$. Panel b): γ generated from the Gaussian distribution with the mean 0 and $\sigma = 0.25\pi$, $\langle\delta\rangle_{1D} \approx -0.19$. Panel c): γ generated from a uniform distribution between 0 and 0.25π , $\langle\delta\rangle_{1D} \approx -0.31$. Panel d): $\gamma = 0$, $\langle\delta\rangle_{1D} \approx -0.42$.

(with initial conditions $D_A = 0$ and $dD_A/dz = 1/H_0$). Thus apart from the background cosmological parameters, the only free parameter left is the mean of density fluctuations along the line of sight $\langle\delta\rangle_{1D}$. Since (9) is just an ordinary differential equation, it is as easy to implement it and solve numerically as the standard relation for the distance in the Friedmann model. The mean of density fluctuations along the line of sight $\langle\delta\rangle_{1D}$ could either be deduced from the galaxy redshift surveys or N-Body simulations.

It should be noted that the analysis presented here was based on the linear approximation (the lensing approximation), i.e. higher order corrections to the distance were not taken into account. It is likely that quadratic corrections may lead to non-negligible δ_D even when $\langle\delta\rangle_{1D} \approx 0$. In this case the presented above modified version of the Dyer–Roeder equation would not be consistent with the actual non-linear distance-redshift relation, as when $\langle\delta\rangle_{1D} = 0$ it reduces to the Friedmannian relation..

The most important conclusion of this paper is that even within the idealized case of large scale homogeneous universe (with only Mpc-scale inhomogeneities) the distance can be different than in the homogeneous universe. The difference can be of order of a few percent. Thus, with an in-

creasing precision of cosmological observations an accurate estimation of the distance is essential. Otherwise errors due to miscalculation the distance can become a major source of systematics.

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APPENDIX A: ALGORITHM OF CALCULATING δ_D FOR MODEL DISCUSSED IN SEC. 4.1

The algorithm of calculating δ_D along the line of sight is as follows

- (i) The radius of a structure R is randomly generated from a uniform distribution, from 0 to 3 Mpc.
- (ii) Then σ_R is calculated using (7). The primordial power spectrum was chosen in agreement with the WMAP7 data (Komatsu et al. 2010): Ak_s^n with $n_s = 0.969$ and the amplitude A chosen so that $\sigma_8 = 0.803$. The transfer function was calculated according to Eisenstein & Hu (1998).
- (iii) An initial value of a density fluctuation δ_0 is generated from the log-normal distribution (6).
- (iv) The evolution of δ (at a fixed point) is calculated using the linear approximations (Peebles 1980)

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = \frac{4\pi G}{c^2}\rho\delta. \quad (\text{A1})$$

- (v) Using the Poisson equation, $\nabla^2\phi$ is calculated and inserted into (5) which is solved from χ_i to $\chi = \chi_i + \chi(2R)$.

- (vi) Steps (i)–(v) are repeated so (5) is solved from $\chi = 0$ to χ_e .

The cosmological parameters are the same as derived from the 7-year WMAP data: $H_0 = 71.4 \text{ km s}^{-1} \text{ Mpc}^{-1}$, $\Omega_m = 0.262$, $\Omega_\Lambda = 0.738$ (Komatsu et al. 2010).

APPENDIX B: ALGORITHM OF CALCULATING δ_D FOR MODEL DISCUSSED IN SEC. 4.2

The algorithm of calculating δ_D is as follows

- (i) First the radius of a void R_v is generated from the Gaussian distribution with the mean of 12 Mpc and the standard deviation of 2 Mpc.
- (ii) Density within the void Ω_v is generated from the Gaussian distribution with the mean $0.2\Omega_m$ and $\sigma = 0.27\Omega_m$. If a generated in this way Ω_v is lower than $0.01\Omega_m$ then the generation is repeated. If after 6 times it is still less than $0.01\Omega_m$ then Ω_v is generated for a uniform distribution between 0 and $0.01\Omega_m$. If $\Omega_v \geq \Omega_m$ then Ω_v is generated one more time. If after 6 times $\Omega_v \geq \Omega_m$ then its value is chosen for a uniform distribution from $0.85\Omega_m$ to Ω_m .
- (iii) Density of the surrounding shell Ω_s is generated from the Gaussian distribution with the mean of $1.75\Omega_m$ and $\sigma = 0.7\Omega_m$. If $\Omega_s \leq \Omega_m$ then its value is generated again. If after 6 times $\Omega_s \leq \Omega_m$ then its value is generated for a uniform distribution between $1.75\Omega_m$ and $1.95\Omega_m$.
- (iv) The condition, that the structure is compensated implies that the radius of the whole structure is

$$R = R_v \left(\frac{\Omega_s - \Omega_v}{\Omega_s - \Omega_m} \right)^{1/3}.$$

- (v) The angle at which the light ray enters the structure is generated using 4 different methods – for details see Sec. 4.2.

- (vi) The evolution of δ (at a fixed point) was calculated using (A1).

- (vii) Integral (5) is solved from χ_i to χ_f (where χ_i is the comoving coordinate of the entry point and χ_f the point where the light ray exits the structure).

- (viii) Steps (i)–(vii) are repeated so (5) is solved from $\chi = 0$ to χ_e .

If instead of the Gaussian, the uniform PDF is used (steps (i)–(iii)) then the PDF of δ is not log-normal PDF as presented in Fig. 3.

The cosmological parameters are the same as derived from the 7-year WMAP data: $H_0 = 71.4 \text{ km s}^{-1} \text{ Mpc}^{-1}$, $\Omega_m = 0.262$, $\Omega_\Lambda = 0.738$ (Komatsu et al. 2010).

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